A GENERALIZATION OF THE ALUTHGE TRANSFORMATION IN THE VIEWPOINT OF MATRIX MEANS (Research on structure of operators using operator means and related topics)

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数理解析研究所講究録 第2113巻 2019年 60-71

A GENERALIZATION OF THE ALUTHGE TRANSFORMATION IN THE VIEWPOINT OF MATRIX MEANS

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Abstract. The Aluthge transformation is generalized in the viewpoint of the axiom of matrix means. It includes the mean transformation which is defined by S. H. Lee, W. Y. Lee and Yoon. Then we shall give some properties of it. Especially, we shall show that n-th iteration of the mean transformation of an invertible matrix converges to a normal matrix. The numerical ranges of these transformations are considered. Throughout this report, we shall consider only n–by–n matrices.

1. Introduction

Let $\mathcal{M}_n$ be a set of all n–by–n matrices. Let $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of T. The Aluthge transformation is defined by $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ in [1]. The Aluthge transformation has been studied by many authors, especially, if T is a bounded linear operator on a complex Hilbert space, then the Aluthge transformation preserves the spectrum, and $\Delta(T)$ has a non-trivial invariant subspace if and only if T has so. Moreover if T is semi-hyponormal (i.e., $|T^*| \leq |T|$), then $\Delta(T)$ is hyponormal (i.e., $\Delta(T)\Delta(T)^* \leq \Delta(T)^*\Delta(T)$). Hence the Aluthge transformation $\Delta(T)$ of an operator is looked like to have better properties than that of T. Recently, new operator transformations were defined in [11, 12] which were similar to the Aluthge transformation. In this paper, we shall unify these transformations in the viewpoint of matrix means, and give some properties in the finite dimensional Hilbert space case.

Matrix mean of positive definite matrices is defined by Kubo-Ando as follows. Let $\mathcal{M}_n^+$ be the cone of positive definite matrices.

Definition 1 (Matrix mean, [10]). *Let $\mathfrak{M} : \mathcal{M}_n^{+2} \to \mathcal{M}_n^+$ be a binary operation. If $\mathfrak{M}$ satisfies the following four conditions, then $\mathfrak{M}$ is called a matrix mean.*

(1) *If $A \leq C$ and $B \leq D$, then $\mathfrak{M}(A, B) \leq \mathfrak{M}(C, D)$,*
(2) *$X^*\mathfrak{M}(A, B)X \leq \mathfrak{M}(X^*AX, X^*BX)$ for all $X \in \mathcal{M}_n$,*

2010 *Mathematics Subject Classification.* Primary 47A64. Secondary 15B48, 47A12, 47A30.

Key words and phrases. The Aluthge transformation; the mean transformation; matrix mean; polar decomposition; double integral transformation.

(3) $\mathfrak{M}$ *is upper semi-continuous on* $\mathcal{M}_n^+$,
(4) $\mathfrak{M}(I, I) = I$, *where* I *means the identity matrix in* $\mathcal{M}_n$.

We notice that if X is invertible in (2), then equality holds. To get a concrete formula of a matrix mean, the following relation is very important.

Theorem A ([10]). *Let* $\mathfrak{M}$ *be a matrix mean. Then there exists a matrix monotone function* f *on* $(0, \infty)$ *such that* $f(1) = 1$ *and*

$$\mathfrak{M}(A, B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$$

for all invertible $A \in \mathcal{M}_n^+$ *and* $B \in \mathcal{M}_n^+$. *A function* f *is called a representing function of a matrix mean* $\mathfrak{M}$.

Especially, if the assumption $f(1) = 1$ is removed, then $\mathfrak{M}(A, B)$ is called a solidarity [5] or a perspective [4]. We note that for a matrix mean $\mathfrak{M}_f$ with a representing function f, $f'(1) = \lambda \in [0, 1]$ (cf. [6]), and we call $\mathfrak{M}_f$ a λ-weighted matrix mean. Typical examples of matrix means are the λ-weighted geometric and λ-weighted power means. These representing functions are $f(x) = x^\lambda$ and $f(x) = [1 - \lambda + \lambda x^t]^{\frac{1}{t}}$, respectively, where $\lambda \in [0, 1]$ and $t \in [-1, 1]$ (in the case $t = 0$, we consider $t \to 0$). The weighted power mean interpolates the arithmetic, the geometric and the harmonic means by putting $t = 1, 0, -1$, respectively. In what follows, the λ-weighted geometric and λ-weighted power means of $A, B \in \mathcal{M}_n^+$ are denoted by $A \sharp_\lambda B$ and $P_t(\lambda; A, B)$, respectively, i.e.,

$$A \sharp_\lambda B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\lambda A^{\frac{1}{2}},$$

$$P_t(\lambda; A, B) = A^{\frac{1}{2}} \left[1 - \lambda + \lambda (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t\right]^{\frac{1}{t}} A^{\frac{1}{2}}.$$

For $A, B \in \mathcal{M}_n$, let $\mathbb{L}_A$ and $\mathbb{R}_B$ be linear maps on $\mathcal{M}_n$ defined as follows:

$$\mathbb{L}_A(X) = AX, \qquad \mathbb{R}_B(X) = XB$$

for $X \in \mathcal{M}_n$. They are called left and right multiplications, respectively. We can consider the arithmetic mean of $\mathbb{L}_A$ and $\mathbb{R}_B$ as follows:

$$\frac{\mathbb{L}_A + \mathbb{R}_B}{2}(X) = \frac{AX + XB}{2}$$

Especially, let $T = U|T|$ be the polar decomposition of an operator T. Put $A = B = |T|$. Then we have

$$\frac{\mathbb{L}_{|T|} + \mathbb{R}_{|T|}}{2}(U) = \frac{|T|U + U|T|}{2}$$

which is called the mean transformation [11]. In this report, we shall discuss generalization of the Aluthge transformation in the viewpoint of matrix means of left and right multiplications.

This report is organized as follows: In Section 2, we shall give a definition of generalized Aluthge transformation in the viewpoint of matrix means of left and right multiplications. Then we shall give some basic properties of the generalized Aluthge transformations. In Section 3, we shall prove that the iteration of mean transformation of an invertible matrix converges to a normal matrix. In Section 4, we shall give inclusion relations of numerical ranges among generalized Aluthge transformations.

2. Generalized Aluthge transformation

In this section, we shall give a definition of the generalized Aluthge transformation. $\mathcal{M}_n$ is a Hilbert space with an inner product $\langle A, B\rangle = \mathrm{tr}B^*A$ for $A, B \in \mathcal{M}_n$. Also for a positive matrix A, $\mathbb{L}_A$ and $\mathbb{R}_A$ are positive definite since

$$\langle \mathbb{L}_A X, X\rangle = \mathrm{tr}X^*AX \geq 0$$

and $\langle \mathbb{R}_A X, X\rangle \geq 0$ for all $X \in \mathcal{M}_n$. Hence we can consider functional calculus of left and right multiplications. It is easy to see that for a positive matrix A, $f(\mathbb{L}_A) = \mathbb{L}_{f(A)}$ and $f(\mathbb{R}_A) = \mathbb{R}_{f(A)}$ hold for any analytic function f defined on $\sigma(A)$, a spectrum of A. Moreover since

$$\mathbb{R}_B\mathbb{L}_A(X) = \mathbb{R}_B(AX) = AXB = \mathbb{L}_A\mathbb{R}_B(X),$$

$\mathbb{L}_A$ and $\mathbb{R}_B$ are commuting. Hence geometric mean of $\mathbb{L}_A$ and $\mathbb{R}_B$ are obtained as follows.

$$\mathbb{L}_A\sharp_{1/2}\mathbb{R}_B(X) = A^{\frac{1}{2}}XB^{\frac{1}{2}}$$

for all $X \in \mathcal{M}_n$. Especially, let $T = U|T|$ be the polar decomposition. then the Aluthge transformation $\Delta(T) := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ can be considered as $\mathbb{L}_{|T|}\sharp_{1/2}\mathbb{R}_{|T|}(U)$, and the mean transformation can be considered as $\frac{\mathbb{L}_{|T|}+\mathbb{R}_{|T|}}{2}(U)$. Here we shall generalize these operator transformation via matrix means.

Definition 2 (Generalized Aluthge transformation)**.** *Let $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible matrix, and let $\mathfrak{M}_f$ be an matrix mean with a representing function f. Then the the generalized Aluthge transformation $\Delta_{\mathfrak{M}_f}(T)$ is defined by the following formula.*

$$\Delta_{\mathfrak{M}_f}(T) := \mathbb{R}_{|T|}f(\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})(U).$$

Especially, let $|T| = V^*\mathrm{diag}(s_1, ..., s_n)V$, where V is a unitary matrix. Then

$$\Delta_{\mathfrak{M}_f}(T) = V^*\left([\mathcal{P}_f(s_j, s_i)] \circ VUV^*\right)V,$$

where $\mathcal{P}_f(a, b) := af(\frac{b}{a})$ a perspective function for $a, b > 0$, and $A \circ B$ means the Hadamard product for $A, B \in \mathcal{M}_n$. We remark that for $a, b > 0$, $\mathcal{P}_f(a, b)I = \mathfrak{M}_f(aI, bI)$.

Proposition 1. *Let $T \in \mathcal{M}_n$ be invertible and $T = U|T|$ be the polar decomposition, and $\mathfrak{M}_f$ be a matrix mean with a representing function f. Then $T = \Delta_{\mathfrak{M}_f}(T)$ if and only if T is normal, i.e., $|T|U = U|T|$.*

Proof. If $U|T| = |T|U$, then $\mathbb{R}_{|T|}(U) = \mathbb{L}_{|T|}(U)$ holds. Let $f(t) = \sum_{n=0}^{\infty} a_n t^n$ be the Taylor expansion. Then we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \mathbb{R}_{|T|} f(\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})(U) \\
&= \mathbb{R}_{|T|}\left(\sum_{n=0}^{\infty} a_n (\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})^n\right)(U) \\
&= \mathbb{R}_{|T|}\left(\sum_{n=0}^{\infty} a_n \mathbb{R}_{|T|}^{-n}\mathbb{L}_{|T|}^{n}\right)(U) \\
&= \mathbb{R}_{|T|}\left(\sum_{n=0}^{\infty} a_n \mathbb{L}_{|T|}^{-n}\mathbb{L}_{|T|}^{n}\right)(U) \\
&= \mathbb{R}_{|T|}\left(\sum_{n=0}^{\infty} a_n \mathbb{I}\right)(U) = \mathbb{R}_{|T|} f(1) U = U|T|,
\end{aligned}$$

where $\mathbb{I}$ means the identity operator on $\mathcal{M}_n$ and the last equality follows from $f(1) = 1$.

Conversely, assume that $T = \Delta_{\mathfrak{M}_f}(T)$. Firstly, we have the following

$$T = \Delta_{\mathfrak{M}_f}(T) = \mathbb{R}_{|T|} f(\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})(U) = f(\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})(T).$$

Since f is an operator monotone function, there exists an inverse function f^{-1}. Let $f^{-1}(t) = \sum_{n=0}^{\infty} b_n t^n$ be the Taylor expansion of f^{-1}. Then we have

$$\begin{aligned}
\mathbb{L}_{|T|}^{-1}\mathbb{R}_{|T|}(T) &= f^{-1}(f(\mathbb{L}_{|T|}^{-1}\mathbb{R}_{|T|}))(T) \\
&= \left(\sum_{n=0}^{\infty} b_n f(\mathbb{L}_{|T|}^{-1}\mathbb{R}_{|T|})^n\right)(T) \\
&= \left(\sum_{n=0}^{\infty} b_n \mathbb{I}\right)(T) = T,
\end{aligned}$$

where the last equality follows from $f^{-1}(1) = 1$. Hence we have $|T|^{-1}T|T| = T$, that is, $|T|U = U|T|$. $\square$

In the case of $f(t) = t^{\frac{1}{2}}$, $\Delta_{\mathfrak{M}_f}$ is the same as the Aluthge transformation, i.e., $\Delta_{\mathfrak{M}_f}(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, and it is well known that $\sigma(\Delta_{\mathfrak{M}_f}(T)) = \sigma(T)$ holds for all operators in $B(\mathcal{H})$. But in the case of other matrix mean, it is not true (cf. [11]). More precisely, we have the following property.

Theorem 1. *Let $T \in \mathcal{M}_n$ be invertible and $\mathfrak{M}_f$ be a matrix mean with a representing function. Then* $\operatorname{trace}(T) = \operatorname{trace}(\Delta_{\mathfrak{M}_f}(T))$.

To prove this, we prepare the following another formula of $\Delta_{\mathfrak{M}_f}(T)$.

Theorem 2. *Let $T \in \mathcal{M}_n$ be invertible, and $\mathfrak{M}_f$ be a matrix mean with a representing function f. Then there exists a positive probability measure μ such that*

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt d\mu(\lambda).$$

Proof. First of all, we shall show Theorem 2 in the harmonic mean case. The harmonic mean case has been considered in [12]. In that paper, the author has considered only weighted shift, and has not obtained any concrete formula. Let $T = U|T|$ be the polar decomposition of T. For $\lambda \in [0,1]$, let $\mathfrak{H}_\lambda$ be a weighted matrix harmonic mean of a weight λ, i.e., the representing function is

$$f(t) = [1 - \lambda + \lambda t^{-1}]^{-1}.$$

We have

$$\begin{aligned}\Delta_{\mathfrak{H}_\lambda}(T) &= \mathbb{R}_{|T|}[(1-\lambda)\mathbb{I} + \lambda(\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})^{-1}]^{-1}(U) \\ &= [(1-\lambda)\mathbb{I} + \lambda(\mathbb{R}_{|T|^{-1}}\mathbb{L}_{|T|})^{-1}]^{-1}(T).\end{aligned}$$

Let $X = \Delta_{\mathfrak{H}_\lambda}(T)$. Then we have

$$\begin{aligned} & X = [(1-\lambda)\mathbb{I} + \lambda(\mathbb{R}_{|T|^{-1}}\mathbb{L}_{|T|})^{-1}]^{-1}(T) \\ \iff & [(1-\lambda)\mathbb{I} + \lambda(\mathbb{R}_{|T|^{-1}}\mathbb{L}_{|T|})^{-1}](X) = U|T| \\ \iff & (1-\lambda)X|T|^{-1} + \lambda|T|^{-1}X = U \\ \iff & X\{(1-\lambda)|T|^{-1}\} - \{-\lambda|T|^{-1}\}X = U.\end{aligned}$$

Here the spectral of $(1-\lambda)|T|^{-1}$ and $-\lambda|T|^{-1}$ are included in the right and left open complex half plane, respectively. So by [7] (cf. [2, Theorem VII.2.3]), we have

$$X = \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt.$$

Hence we take a probability measure μ as $\mu(1) = 1$, and obtain Theorem 2 in the harmonic mean case.

Next, we shall show Theorem 2 in other matrix mean case. Let $\mathfrak{M}_f$ be a matrix mean with a representing function f. Then it is known that (cf. [6]) there exists a positive probability measure μ such that

$$f(t) = \int_0^1 [1 - \lambda + \lambda t^{-1}]^{-1} d\mu(\lambda).$$

Hence we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \mathbb{R}_{|T|} f(\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})(U) \\
&= \mathbb{R}_{|T|} \int_0^1 [(1-\lambda)\mathbb{I} + \lambda(\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})^{-1}]^{-1} d\mu(\lambda)(U) \\
&= \int_0^1 \mathbb{R}_{|T|}[(1-\lambda)\mathbb{I} + \lambda(\mathbb{R}_{|T|}^{-1}\mathbb{L}_{|T|})^{-1}]^{-1}(U) d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt d\mu(\lambda).
\end{aligned}$$

□

Proof of Theorem 1. By Theorem 2, we have

$$\begin{aligned}
\operatorname{trace}\left(\Delta_{\mathfrak{M}_f}(T)\right) &= \operatorname{trace}\left(\int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt d\mu(\lambda)\right) \\
&= \operatorname{trace}\left(\int_0^1 \int_0^\infty U e^{-t|T|^{-1}} dt\right) = \operatorname{trace}(T).
\end{aligned}$$

□

3. Iteration of the mean transformation

In this section, we shall prove that the iteration of the mean transformation of an invertible matrix (i.e., the generalized Aluthge transformation in the case of non-weighted arithmetic mean) converges to a normal matrix. Iteration of mean transformation has been considered in [3]. In that paper, the authors give a complete form of the limit of iterated mean transformation of rank one operator.

Theorem 3. *Let $T = U|T|$ be the polar decomposition of an invertible matrix T, and let $\mathfrak{A}$ be a non-weighted arithmetic mean. Then the sequence $\{\Delta_{\mathfrak{A}}^n(T)\}$ converges to a normal matrix N, where $\Delta_{\mathfrak{A}}^n(T) := \Delta_{\mathfrak{A}}(\Delta_{\mathfrak{A}}^{n-1}(T))$. It satisfies* $\operatorname{trace}(T) = \operatorname{trace}(N)$ *and* $\operatorname{trace}(|T|) = \operatorname{trace}(|N|)$.

Proof. First of all, we shall give a polar decomposition of $\Delta_{\mathfrak{A}}^n(T)$ ($n = 1, 2, ...$) by induction on n. Assume that let $\Delta_{\mathfrak{A}}^n(T) = U_n P_n$ be the polar decomposition of $\Delta_{\mathfrak{A}}^n(T)$ with a partial isometry U_n and a positive matrix P_n. Since U is unitary,

$$\Delta_{\mathfrak{A}}(T) = \frac{\mathbb{R}_{|T|} + \mathbb{L}_{|T|}}{2}(U) = \frac{U|T| + |T|U}{2} = U\frac{|T| + U^*|T|U}{2}$$

and

$$|\Delta_{\mathfrak{A}}(T)|^2 = \frac{|T| + U^*|T|U}{2} U^*U \frac{|T| + U^*|T|U}{2} = \left(\frac{|T| + U^*|T|U}{2}\right)^2.$$

Since $\frac{|T|+U^*|T|U}{2} \geq 0$, $U_1 = U$ and $P_1 = \frac{|T|+U^*|T|U}{2}$.

Assume that

$$U_j = U \text{ and } P_j = \frac{1}{2^j}\sum_{k=0}^{j}\binom{j}{k} U^{*k}|T|U^k$$

for all $j = 1, 2, ..., n-1$. We have

$$\begin{aligned}
\Delta_{\mathfrak{A}}^n(T) &= \frac{U_{n-1}P_{n-1} + P_{n-1}U_{n-1}}{2} \\
&= \frac{UP_{n-1} + P_{n-1}U}{2} \\
&= U\frac{P_{n-1} + U^*P_{n-1}U}{2} \\
&= U\frac{1}{2}\left(\frac{1}{2^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k} U^{*k}|T|U^k + U^*\left[\frac{1}{2^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k} U^{*k}|T|U^k\right]U\right) \\
&= U\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} U^{*k}|T|U^k.
\end{aligned}$$

Since $P_n = \frac{P_{n-1}+U^*P_{n-1}U}{2}$, we have $U_n = U$ and $P_n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} U^{*k}|T|U^k$.

By Theorem 1 and the above, we have $\mathrm{trace}(T) = \mathrm{trace}(\Delta_{\mathfrak{A}}^n(T))$ and $\mathrm{trace}(|T|) = \mathrm{trace}(|\Delta_{\mathfrak{A}}^n(T)|)$ for all $n = 0, 1, 2, ...$ Hence we shall only prove that $\{|\Delta_{\mathfrak{A}}^n(T)|\}$ converges. Since U is unitary, we can diagonalize U as $U = V^*DV$ with a unitary matrix V and $D = \mathrm{diag}(e^{\theta_1\sqrt{-1}}, ..., e^{\theta_n\sqrt{-1}})$. Then

$$\begin{aligned}
\Delta_{\mathfrak{A}}(T) &= \frac{U|T| + |T|U}{2} \\
&= \frac{V^*DV|T|V^*V + V^*V|T|V^*DV}{2} \\
&= V^*\frac{DV|T|V^* + V|T|V^*D}{2}V = V^*\Delta_{\mathfrak{A}}(DV|T|V^*)V.
\end{aligned}$$

We notice that for every $T \in \mathcal{M}_n$ and unitary U, we have

$$\Delta_{\mathfrak{A}}(U^*TU) = U^*\Delta_{\mathfrak{A}}(T)U.$$

Hence $\Delta_{\mathfrak{A}}^n(T) = V^*\Delta_{\mathfrak{A}}^n(DV|T|V^*)V$, and

$$\begin{aligned}
|\Delta_{\mathfrak{A}}^n(T)| &= V^*|\Delta_{\mathfrak{A}}^n(DV|T|V^*)|V \\
&= V^*\left\{\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} D^{*k}V|T|V^*D^k\right\}V.
\end{aligned}$$

Let $V|T|V^* = P$. Then $D^{*k}V|T|V^*D^k = P \circ [e^{k(\theta_j-\theta_i)\sqrt{-1}}]$ and

$$\begin{aligned}\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} D^{*k}V|T|V^*D^k &= \frac{1}{2^n}\left[\sum_{k=0}^{n}\binom{n}{k} e^{k(\theta_j-\theta_i)\sqrt{-1}}\right]\circ P \\ &= \frac{1}{2^n}\left[(1+e^{(\theta_j-\theta_i)\sqrt{-1}})^n\right]\circ P \\ &= \left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^n\right]\circ P.\end{aligned}$$

We notice that $\left|\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right| < 1$ if $\theta_j \neq \theta_i + 2m\pi$ for all integer m, and $\left|\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right| = 1$ if $\theta_j = \theta_i + 2m\pi$ for some integers m. Hence there exists a matrix P_0 such that

$$\lim_{n\to\infty}|\Delta^n_{\mathfrak{A}}(T)| = \lim_{n\to\infty} V^*\left(\left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^n\right]\circ P\right)V = P_0,$$

and we have $\lim_{n\to\infty}\Delta^n_{\mathfrak{A}}(T) = \lim_{n\to\infty}U|\Delta^n_{\mathfrak{A}}(T)| = UP_0$. Moreover $UP_0 = \Delta^n_{\mathfrak{A}}(UP_0) = \frac{UP_0+P_0U}{2}$ holds, and UP_0 is a normal matrix since $UP_0 = P_0U$ holds. □

4. Numerical ranges of the generalized Aluthge transformations

In this section, we shall show inclusion relations among numerical ranges of the generalized Aluthge transformations. We shall consider a relation $\preceq$ between two matrix means. It is defined as follows. Let $\mathfrak{M}_f$ and $\mathfrak{N}_g$ be matrix means with representing functions f and g, respectively. $\mathfrak{M}_f \preceq \mathfrak{N}_g$ if and only if for any natural number m and $s_i > 0$ $(i = 1, 2, ..., m)$

$$\left[\frac{\mathcal{P}_f(s_i,s_j)}{\mathcal{P}_g(s_i,s_j)}\right]$$

is a positive definite matrix. It is known that Let $\mathfrak{A}$, $\mathfrak{L}$, $\mathfrak{G}$ and $\mathfrak{H}$ be non-weighted arithmetic, logarithmic, geometric and harmonic means. Then

$$\mathfrak{H} \preceq \mathfrak{G} \preceq \mathfrak{L} \preceq \mathfrak{A}$$

(cf. [8]). In this section, we shall consider only symmetric matrix means. A matrix mean $\mathfrak{M}_f$ is symmetric if and only if $\mathfrak{M}_f(A, B) = \mathfrak{M}_f(B, A)$.

Theorem 4. *Let $T \in \mathcal{M}_n$, and let $\mathfrak{M}_f, \mathfrak{N}_g$ be symmetric matrix means with representing functions f and g, respectively. Then the following hold.*

(1) *if $\mathfrak{M}_f \preceq \mathfrak{N}_g$, then $\overline{W(\Delta_{\mathfrak{M}_f}(T))} \subseteq \overline{W(\Delta_{\mathfrak{N}_g}(T))}$ holds,*

(2) *if $\mathfrak{M}_f \preceq \mathfrak{A}$, then $\overline{W((\Delta_{\mathfrak{M}_f}(T))} \subseteq \overline{W(T)}$ holds,*

where $\mathfrak{A}$ is the non-weighted arithmetic mean.

To prove Theorem 4, we shall use an idea of double integral transformation for the generalized Aluthge transformation.

Theorem 5. *Let $T \in \mathcal{M}_n$ be invertible, and let $T = U|T|$ be the polar decomposition of T. Assume that $|T| = \sum_{k=1}^n s_k P_k$ is a spectral decomposition of a positive invertible matrix $|T|$. Then for a matrix mean $\mathfrak{M}_f$ with a representing function f,*

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{k,l=1}^{n} \mathcal{P}_f(s_k, s_l) P_k U P_l.$$

Proof. First of all, for a representing function f of a matrix mean $\mathfrak{M}_f$, there exists a positive probability measure μ, s.t.,

$$f(t) = \int_0^1 [1 - \lambda + \lambda t^{-1}]^{-1} d\mu(\lambda).$$

Then the perspective function of f is obtained as follows:

$$\mathcal{P}_f(a, b) = af\left(\frac{b}{a}\right) = \int_0^1 [(1-\lambda)a^{-1} + \lambda b^{-1}]^{-1} d\mu(\lambda) \tag{1}$$

for $a, b > 0$. By Theorem 2, we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty \left(\sum_{k=1}^n e^{-(1-\lambda)ts_k^{-1}} P_k\right) U \left(\sum_{l=1}^n e^{-\lambda t s_l^{-1}} P_l\right) dt d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty \sum_{k,l=1}^n e^{-t[(1-\lambda)s_k^{-1} + \lambda s_l^{-1}]} P_k U P_l dt d\mu(\lambda) \\
&= \sum_{k,l=1}^n \int_0^1 \int_0^\infty e^{-t[(1-\lambda)s_k^{-1} + \lambda s_l^{-1}]} dt d\mu(\lambda) P_k U P_l \\
&= \sum_{k,l=1}^n \int_0^1 [(1-\lambda)s_k^{-1} + \lambda s_l^{-1}]^{-1} d\mu(\lambda) P_k U P_l \\
&= \sum_{k,l=1}^n \mathcal{P}_f(s_k, s_l) P_k U P_l,
\end{aligned}$$

where the last equality holds by (1). □

Theorem 5 can be considered as matrix version of a double integral transformation which has been introduced in [8]. In [8], it has not been shown whether every operator mean can be represented to the double integral transformation in the infinite Hilbert space case or not. However, matrix case is true as in Theorem 5.

In the case of infinite dimensional case, let $|T| = \int_0^{\|T\|} sdE_s$ be a spectral decomposition, if an operator mean $\mathfrak{M}_f$ can be represented to the form of double integral transformation, then it is given as follows:

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^{\|T\|}\int_0^{\|T\|} \mathcal{P}_f(s,t)dE_sUdE_t.$$

In [8], they discussed more general case. Let $H, K \in B(\mathcal{H})$ be positive operators with the spectral decompositions are

$$H = \int_0^{\|H\|} sdE_s \text{ and } F = \int_0^{\|F\|} tdF_t,$$

respectively. Then for an operator mean $\mathfrak{M}_f$ with a representing function f,

$$\mathfrak{M}_f(H,K)(X) := \int_0^{\|H\|} \mathcal{P}_f(s,t)dE_sXdF_t$$

for $X \in B(\mathcal{H})$ if it exists. Especially, if $H, K \in \mathcal{M}_n^+$ with the spectral decompositions

$$H = \sum_{k=1}^n s_kP_k \text{ and } K = \sum_{l=1}^n t_lQ_l,$$

then

$$\mathfrak{M}_f(H,K)(X) := \sum_{k,l=1}^n \mathcal{P}_f(s_k,t_l)P_kXQ_l$$

for $X \in \mathcal{M}_n$. The following norm inequality is important.

Theorem 6 ([8])**.** *Let $\mathfrak{M}, \mathfrak{N}$ be operator means ($\mathfrak{M} \preceq \mathfrak{N}$) and H, K be positive operators. Then for any unitarily invariant norm $|||\cdot|||$, we have*

$$|||\mathfrak{M}(H,K)X||| \leq |||\mathfrak{N}(H,K)X|||$$

for all $X \in B(\mathcal{H})$.

Moreover, we shall prepare the following result to prove Theorem 4.

Theorem 7 ([9, 13])**.** *Let $T \in B(\mathcal{H})$. Then*

$$\overline{W(T)} = \bigcap_{\lambda\in\mathbb{C}} \{\mu \ : \ |\mu - \lambda| \leq \|T - \lambda I\|\}.$$

Using the above results, we shall show Theorem 4.

Proof of Theorem 4. (1) Let $|T| = \sum_{k=1}^n s_kP_k$ be the spectral decomposition of $|T|$. Then $|T|$ and each projection P_i are commuting, and

$P_iP_j = O$ for all $i \neq j$. Then for $\lambda \in \mathbb{C}$,

$$\begin{aligned}
&\mathfrak{M}_f(|T|,|T|)(U-\lambda|T|^{-1})\\
&=\sum_{k,l=1}^{n}\mathcal{P}_f(s_k,s_l)P_k(U-\lambda|T|^{-1})P_l\\
&=\sum_{k,l=1}^{n}\mathcal{P}_f(s_k,s_l)P_kUP_l-\lambda\sum_{k,l=1}^{n}\mathcal{P}_f(s_k,s_l)P_k|T|^{-1}P_l\\
&=\sum_{k,l=1}^{n}\mathcal{P}_f(s_k,s_l)P_kUP_l-\lambda\sum_{l=1}^{n}\mathcal{P}_f(s_l,s_l)P_l|T|^{-1}\\
&=\Delta_{\mathfrak{M}_f}(T)-\lambda I,
\end{aligned}$$

where the last equality follows from $\mathcal{P}_f(a,a) = af(\frac{a}{a}) = a$. Then by Theorem 6, we have

$$\begin{aligned}
\|\Delta_{\mathfrak{M}_f}(T)-\lambda I\| &= \|\mathfrak{M}_f(|T|,|T|)(U-\lambda|T|^{-1})\|\\
&\leq \|\mathfrak{N}_g(|T|,|T|)(U-\lambda|T|^{-1})\| = \|\Delta_{\mathfrak{N}_g}(T)-\lambda I\|
\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Hence by Theorem 7, we have $\overline{W(\Delta_{\mathfrak{M}_f}(T))} \subseteq \overline{W(\Delta_{\mathfrak{N}_g}(T))}$.

(2) Let $\mathfrak{A}$ be a non-weighted arithmetic mean. Then we can show $\overline{W(\Delta_{\mathfrak{A}}(T))} \subseteq \overline{W(T)}$. In fact, $\Delta_{\mathfrak{A}}(T) = \frac{U|T|+|T|U}{2}$ and $W(T) = W(|T|U)$ holds for any invertible matrix T. Hence by (1), we can prove (2). □

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